# Problem 16.4

Show that if we make the change of variables $\xi = x - ct$ and $\eta = x + ct$, then, as in (16.7),

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = -4c^2 \frac{\partial}{\partial \xi} \frac{\partial u}{\partial \eta}.$$

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### Solution

The wave equation is given by equation (16.4) on page 684.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{16.4}$$

Write it in standard form.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Comparing this equation with the general form of a second-order PDE,

$$A \frac{\partial^2 u}{\partial t^2} + B \frac{\partial^2 u}{\partial t \partial x} + C \frac{\partial^2 u}{\partial x^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G,$$

we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dx}{dt} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$

$$\frac{dx}{dt} = \frac{1}{2}\left(0 \pm \sqrt{0 - 4(1)(-c^2)}\right)$$

$$\frac{dx}{dt} = \frac{1}{2}(\pm 2c)$$

$$\frac{dx}{dt} = c \quad \text{or} \quad \frac{dx}{dt} = -c.$$

Note that the discriminant, $B^2 - 4AC = 0 - 4(1)(-c^2) = 4c^2$, is greater than 0, which means that the wave equation is hyperbolic. Therefore, there are two real and distinct families of characteristic curves in the $tx$-plane:

$$x = ct + C_1 \quad \text{or} \quad x = -ct + C_2.$$

Solve for the constants of integration.

$$C_1 = x - ct$$

$$C_2 = x + ct$$

Make the change of variables,

$$\xi = x - ct$$

$$\eta = x + ct,$$

so that the wave equation takes the simplest form. Use the chain rule to write the derivatives in terms of these new variables.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial u}{\partial \xi}(-c) + \frac{\partial u}{\partial \eta}(c) = -c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \left(\frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial t}\right) = \left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)\left(-c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}\right) \\ &= -c\frac{\partial}{\partial \xi}\left(-c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}\right) + c\frac{\partial}{\partial \eta}\left(-c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}\right) = c^2\frac{\partial^2 u}{\partial \xi^2} - c^2\frac{\partial^2 u}{\partial \xi \partial \eta} - c^2\frac{\partial^2 u}{\partial \eta \partial \xi} + c^2\frac{\partial^2 u}{\partial \eta^2} \\ &= c^2\frac{\partial^2 u}{\partial \xi^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \eta^2}\end{aligned}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi}(1) + \frac{\partial u}{\partial \eta}(1) = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) \\ &= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2} \\ &= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\end{aligned}$$

As a result,

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} &= \left(c^2\frac{\partial^2 u}{\partial \xi^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \eta^2}\right) - c^2\left(\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right) \\ &= \cancel{c^2\frac{\partial^2 u}{\partial \xi^2}} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + \xcancel{c^2\frac{\partial^2 u}{\partial \eta^2}} - \cancel{c^2\frac{\partial^2 u}{\partial \xi^2}} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} - \xcancel{c^2\frac{\partial^2 u}{\partial \eta^2}} \\ &= -4c^2\frac{\partial^2 u}{\partial \xi \partial \eta} \\ &= -4c^2\frac{\partial}{\partial \xi}\frac{\partial u}{\partial \eta},\end{aligned}$$

and the wave equation becomes

$$-4c^2\frac{\partial}{\partial \xi}\frac{\partial u}{\partial \eta} = 0.$$

Divide both sides by $-4c^2$.

$$\frac{\partial}{\partial \xi}\frac{\partial u}{\partial \eta} = 0$$

Integrate both sides partially with respect to $\xi$.

$$\frac{\partial u}{\partial \eta} = g(\eta)$$

Integrate both sides partially with respect to $\eta$.

$$u(\xi, \eta) = \int g(\eta)\, d\eta + F(\xi)$$

The integral of an arbitrary function is another arbitrary function.

$$u(\xi, \eta) = G(\eta) + F(\xi)$$

Now that the general solution for $u$ is known, change back to the original variables.

$$u(x, t) = G(x + ct) + F(x - ct)$$